Power operations in Morava E-theory

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Abstract

This talk contains two parts. The first one is a modular interpretation of power operations in Morava E-theory, for which the main reference is [Rez09]. The second one is an application of power operations in Morava E-theory in the computation of homotopy groups, for which the main reference is [BSSW23]. We will also give a short introduction to [BSSW23].

Contents

1 Notation

In this note, all formal groups will be 1-dimensional. We fix a prime number p and $n \in \mathbb{Z}_{>0}$, a perfect field k of characteristic p and a formal group G_0 over k of height n. Let $K(n)$ be the Morava K-theory of height n at p.

Let $E := E(k, G_0)$ be the Morava E-theory. In particular, E is a commutative algebra in $K(n)$-local spectra and $\mathbb{G}_n := \mathrm{Aut}(E)$ is a static group, called ***Morava stabilizer group***. In addition, we have $E^0 \simeq W[\![u_1, \cdots, u_{n-1}]\!]$ non-canonically, where W is the ring of p-typical Witt vectors of k. Let $\mathfrak{m}$ be the maximal ideal of E^0.

2 Background in power operations

Remark 2.1. *There is bad news at the beginning. We cannot follow the definition given in Erik's talk, as the endomorphism $\mathbb{E}_1$-algebra in spectra of the forgetful functor*

$$G\colon \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp})) \to \mathrm{Sp}$$

The point is that, what Erik defined, is the $\mathbb{E}_1$-algebra of stable power operations, as in [GL20]. These are power operations compatible with the suspension. Due to [GL20, Corollary 8.4 and Theorem 8.5], the spectrum of stable power operations at a fixed weight involves a Tate construction. However, Tate spectra vanish $K(n)$-locally because of the ambidexterity in $K(n)$-local spectra (cf. [HL13, Theorem 0.0.1 or more generally Theorem 5.2.1]). Therefore, there are no stable power operations for Morava E-theory, so we have to consider unstable operations. The good news is that we do not need to change much.

We let $G\colon \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp})) \to \mathrm{Sp}$ be the forgetful functor and $j_*\colon \mathrm{Sp}_{K(n)} \to \mathrm{Sp}$ be the inclusion functor. The functor j_* admits a left adjoint j^* and

$$\mathrm{Sp}_{K(n)} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Sp}$$

promotes to a symmetric monoidal adjunction. It induces an adjunction

$$\mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp_{K(n)}})) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp}))$$

The $K(n)$-localization functor $L_{K(n)} \simeq j_* j^*$.

Define U to be the composition

$$U\colon \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp_{K(n)}})) \xrightarrow{\mathrm{j_*}} \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp})) \xrightarrow{\mathrm{G}} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathrm{An}$$

and $\widehat{\mathrm{Pow}}(E) := \mathrm{End}(U)$ be the $\mathbb{E}_1$-algebra in anima of endomorphisms of U (cf. [Lur17, Corollary 4.7.1.40]).

Note that we have the following pairs of adjoint functors.

$$\mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp_{K(n)}})) \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp})) \underset{G}{\overset{\mathrm{Sym}_E}{\rightleftarrows}} \mathrm{Sp} \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathrm{An}$$

By moving right adjoints to the left through the adjunctions,

$$U(R) = \Omega^\infty \circ G \circ j_*(R) \simeq \mathrm{Map}(1, \Omega^\infty \circ G \circ j_*(R)) \simeq \mathrm{Map}(j^*\mathrm{Sym}_E(\mathbb{S}), R)$$

By Yoneda lemma, we have

$$\widehat{\mathrm{Pow}}(E) \simeq \mathrm{Map}(j^*\mathrm{Sym}_E(\mathbb{S}), j^*\mathrm{Sym}_E(\mathbb{S})) \simeq U(j^*\mathrm{Sym}_E(\mathbb{S}))$$

Thus, $\pi_*(\widehat{\mathrm{Pow}}(E)) \simeq \pi_*(j_*j^*\mathrm{Sym}_E(\mathbb{S}))$.

Definition 2.2 (Completed E-homology)**.** For $X \in \mathrm{Sp}$, let the ***completed E-homology*** be $E^\wedge_*(X) := \pi_*(j_*j^*(E \otimes X))$.

For $k \in \mathbb{N}$, let $\mathrm{Sym}^k(X) := (X^{\otimes k})_{h\Sigma_k}$, so $\mathrm{Sym}(X) \simeq \oplus_{k=0}^\infty \mathrm{Sym}^k(X)$. The localization functor does not preserve infinite direct sum. However, we have the following proposition.

Proposition 2.3 (cf. [Rez09, 4.17])**.** $E^\wedge_*(\mathrm{Sym}(\mathbb{S})) \cong \left(\bigoplus_{k=0}^\infty E^\wedge_*(\mathrm{Sym}^k(\mathbb{S}))\right)^\wedge_m$.

This inspires the following definition.

Definition 2.4 (Power operation)**.** For $k \in \mathbb{N}$, the abelian group of ***power operations of E of weight k*** is $E^\wedge_*(B\Sigma_k)$.

Definition 2.5 (Total and additive power operation)**.** Suppose $R \in \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp_{K(n)}}))$. The action of $\widehat{\mathrm{Pow}}(E)$ on $U(R)$ gives us a map

$$\begin{aligned} U(R) &\to \mathrm{Map}(\mathrm{End}(U), U(R)) \simeq \mathrm{Map}(\Omega^\infty G\mathrm{Sym}_E(\mathbb{S}), \Omega^\infty G j_*(R)) \\ &\to \mathrm{Map}(\Omega^\infty(\Sigma^\infty_+ B\Sigma_k \otimes E), \Omega^\infty G j_*(R)) \simeq \mathrm{Map}(\Sigma^\infty_+(\Omega^\infty(\Sigma^\infty_+ B\Sigma_k \otimes E)), G j_*(R)) \\ &\to \mathrm{Map}(\Sigma^\infty_+ B\Sigma_k, G j_*(R)) \end{aligned}$$

Taking π_0 gives us the ***total power operation of*** E ***of weight*** k ***for*** R is $P_k\colon \pi_0(R) \to R^0(B\Sigma_k)$. Explicitly, P_k sends $\alpha\colon \mathbb{S} \to j_*R$ to the composition

$$j^*(\Sigma^\infty_+ B\Sigma_k \otimes E) \simeq j^*\mathrm{Sym}^k_E(\mathbb{S}) \xrightarrow{j^*\mathrm{Sym}^k_E(\alpha)} j^*\mathrm{Sym}^k_E(j_*R) \xrightarrow{j^*\gamma_k} j^*j_*R \simeq R,$$

where γ_k is the multiplication map on j_*R. By pre-composing a power operation of weight k $f \in E^\wedge_r(B\Sigma_k)$, we get a map $f^* \circ P_k\colon \pi_0(R) \to \pi_r(R)$ sending $\alpha\colon \mathbb{S} \to j_*R$ to the composition

$$j^*\Sigma^r E \xrightarrow{f} j^*(\Sigma^\infty_+ B\Sigma_k \otimes E) \simeq j^*\mathrm{Sym}^k_E(\mathbb{S}) \xrightarrow{j^*\mathrm{Sym}^k_E(\alpha)} j^*\mathrm{Sym}^k_E(j_*R) \xrightarrow{j^*\gamma_k} j^*j_*R \simeq R$$

Note that this map is not additive since Sym^k_E is not additive. We call $f \in E^\wedge_r(\mathrm{Sym}^k_E(\mathbb{S}))$ an ***additive power operation of*** E ***of weight*** k if $f^* \circ P_k$ is additive for all R.

Remark 2.6. *Note that* $\widehat{\mathrm{Pow}}(E)$ *is an* $\mathbb{E}_1$*-algebra in anima and* $\pi_*(\widehat{\mathrm{Pow}}(E))$ *also has addition intertwining with the multiplication. This looks like the structure of a plethory, which is a ring object in* Ring^{op}*. In fact, [Rez09, §4] defines an algebraic approximation functor* $T\colon \mathrm{Mod}^\heartsuit_{\mathrm{E}_*} \to \mathrm{Mod}^\heartsuit_{\mathrm{E}_*}$ *such that*

1. *The functor* T *promotes to a monad.*
2. *The forgetful functor* $\mathrm{Alg}_T \to \mathrm{Alg}_{E^*}$ *is plethyistic. In particular,* $T(E_*)$ *is an* E_**-plethory.*
3. *For a flat* E*-module* M*,* $(T(\pi_*(M)))^\wedge_{\mathfrak{m}} \simeq \pi_*(j_*j^*\mathrm{Sym}_E(M))$*, which gives* T *the name algebraic approximation functor.*
4. *If* M *is further finite free, then* $T(\pi_*(M)) \simeq \oplus_{k=0}^\infty \pi_*(j_*j^*\mathrm{Sym}^k_E(M))$*. In particular, we have Proposition 2.3 combining the last point.*

One may also be interested in the beginning of [BSY22, §3] for a short introduction to the case at degree 0*. Under the view of plethory, additive power operations are exactly the primitive elements with respect to the coaddition structure on* $T(E_*)$ *(cf. [Rez09, 6.2]).*

It is not clear to me whether $\pi_*(\widehat{\mathrm{Pow}}(E))$ *is an* E_**-plethory itself.*

The following proposition gives an example of non-additive power operation.

Proposition 2.7. *Let* $i\colon E \to \Sigma^\infty_+ B\Sigma_k \otimes E \in E_0(B\Sigma_k)$ *be the map induced by the canonical map* $\mathbb{S} \to \Sigma^\infty_+ B\Sigma_k$*. We have* $i^* \circ P_k(\alpha) = \alpha^k$*.*

Proof. Note that we have the following commutative diagram.

$$\begin{array}{ccccccc} E\otimes\mathbb{S} & \xrightarrow{\delta} & E\otimes\mathbb{S}^{\otimes k} & \xrightarrow{\alpha^{\otimes k}} & R^{\otimes k} & \longrightarrow & R \\ & \searrow^{i} & \downarrow & & \downarrow & \nearrow_{\gamma_k} & \\ & & \mathrm{Sym}^k_E(\mathbb{S}) & \xrightarrow{\mathrm{Sym}^k_E(\alpha)} & \mathrm{Sym}^k_E(R) & & \end{array}$$

where the composition up is k-th power of α and the composition below is $i^*\circ P_k(\alpha)$. □

Let $\mathrm{tr}^{\Sigma_k}_{\Sigma_i\times\Sigma_{k-i}}$ be the transfer map defined in Appendix A. Define

$$\mathrm{Tr}_i\colon R^0\big(B\Sigma_i\otimes B\Sigma_{k-i}\big)\xrightarrow{(\mathrm{tr}^{\Sigma_k}_{\Sigma_i\times\Sigma_{k-i}})^*} R^0(B\Sigma_k)$$

and I_{tr} be the transfer ideal in $R^0(B\Sigma_k)$ generated by the images of Tr_i for $0<i<k$. By [BMMS86, II.2.1], we have

$$P_k(\alpha+\beta)=P_k(\alpha)+P_k(\beta)+\sum_{0<i<k}\mathrm{Tr}_i\big(P_i(\alpha)P_{k-i}(\beta)\big)$$

Proposition 2.8. *The composition*

$$\overline{P_k}\colon R^0(\mathbb{S})\xrightarrow{P_k} R^0(B\Sigma_k)\to R^0(B\Sigma_k)/I_{tr}$$

is a ring homomorphism, called the **total additive power operation of** E.

If $f\in E^\wedge_0(B\Sigma_k)$, then f is additive if and only if $f^*\circ\mathrm{Tr}_i=0$ for all $0<i<k$, or equivalently, if $f^*\colon R^0(B\Sigma_k)\to\pi_0(R)$ factors through $R^0(B\Sigma_k)/I_{tr}$.

The localization functor buys us the following benefit of preserving finite freeness.

Proposition 2.9 ([Rez09, 3.17])**.** *If $M\in\mathrm{Mod}_{\mathrm{E}}$ is a finite free E-module, then $L_{K(n)}\mathrm{Sym}^k_E(M)$ is also finite free for all k. Furthermore, if $\pi_*(M)$ concentrate on even degrees, then so is $L_{K(n)}\mathrm{Sym}^k_E(M)$.*

In particular, $E^\wedge_*(B\Sigma_k)$ concentrates on even degrees. Since E is even periodic, we may assume that $r=0$ in the following. For careful consideration of the degree issues, see [Rez09, §2, §7 and 11.17].

Corollary 2.10. $\mathrm{Hom}_{E^0}\big(E^\wedge_0(B\Sigma_k),E^0\big)\simeq E^0(B\Sigma_k)$*, equivalently,* $\mathrm{Hom}_{E^0}\big(E^0(B\Sigma_k),E^0\big)\simeq E^\wedge_0(B\Sigma_k)$.

Therefore, the module Γ_k of additive power operations in $E_0^\wedge(B\Sigma_k)$ is isomorphic to the dual module $\mathrm{Hom}_{E^0}\big(E^0(B\Sigma_k)/I_{tr}, E^0\big)$. Let $A_k \simeq E^0(B\Sigma_k)/I_{tr}$ be the dual module, so A_k carries a canonical E^0-algebra structure. Note that by composition, $\Gamma := \oplus_{k=0}^\infty \Gamma_k$ is a graded $\mathbb{E}_1$-algebra in $\mathrm{Mod}_{\mathrm{E}^0}^\heartsuit$. For each commutative complete E-algebra R, $\pi_0(R)$ is a (static) left Γ-module.

Proposition 2.11. *If k is not a power of p, then $I_{tr} = E^0(B\Sigma_k)$.*

Proof. If k is not a power of p, then there is an $i < k$ such that $\binom{k}{i}$ is coprime to p. Since $E^0(B\Sigma_k)$ is p-local, the composition $\Sigma_+^\infty B\Sigma_k \xrightarrow{\mathrm{tr}_{\Sigma_i\times\Sigma_{k-i}}^{\Sigma_k}} \Sigma_+^\infty(B\Sigma_i \otimes B\Sigma_{k-i}) \to \Sigma_+^\infty B\Sigma_k$ is an isomorphism in E-cohomology by Proposition A.1, so the image of the transfer map is the whole ring. □

Thus, $A_k = 0$ for k is not a power of p, and we denote $A[r] := A_{p^r}$.

Recall from Xiansheng's talk that we have the following interpretation of homotopy groups of spectra in terms of quasi-coherent sheaves over the moduli stack of formal groups.

Proposition 2.12. *There is an equivalence of symmetric monoidal abelian categories*

$$\mathrm{QCoh}(\mathcal{M}_{fg}^{\mathrm{Lie}\simeq\mathrm{triv}})^\heartsuit \simeq \lim_{\Delta} \mathrm{QCoh}(\mathrm{Spec}(\pi_*(\mathrm{MU}^{\otimes[-]})))$$

Furthermore, the MU*-homology promotes to a functor*

$$\mathrm{Sp} \to \lim_{\Delta} \mathrm{QCoh}(\mathrm{Spec}(\pi_*(\mathrm{MU}^{\otimes[-]})))^\heartsuit \simeq \mathrm{QCoh}(\mathcal{M}_{fg}^{\mathrm{Lie}\simeq\mathrm{triv}})^\heartsuit$$

Out first goal is to understand the left Γ-module structure of the homotopy groups of commutative $K(n)$-local E-algebras in terms of deformations of formal groups.

3 Deformation of Frobenius

Let $\widehat{\mathcal{R}}$ be the category of complete Noetherian local rings, whose residue field is an extension of k, and local homomorphisms. From now on, we will abuse the notion of a morphism in $\widehat{\mathcal{R}}$ with its corresponding morphism between affine formal schemes.

Definition 3.1 (Deformation)**.** A ***deformation*** of G_0 to $R \in \widehat{\mathcal{R}}$ is a triple (H, i, α), consisting of

1. a formal group H over R,

2. a homomorphism $i\colon k \to R/\mathfrak{n}$, where $\mathfrak{n}$ is the maximal ideal of R and

3. an isomorphism $\alpha\colon H_0 \to \widetilde{G_0}$, where H_0 is the special fiber of H and $\widetilde{G_0}$ is the base-change of G_0 along i.

Definition 3.2 (Frobenius isogeny)**.** Suppose $T \in \widehat{\mathcal{R}}$ has characteristic p. Let $\phi\colon T \to T$ be the Frobenius homomorphism. For any formal group H/T, the ***Frobenius isogeny*** $\mathrm{Frob}\colon H \to H^{(p)}$ is the homomorphism of formal groups over R induced by the following diagram.

$$\begin{array}{ccccc} & & \overset{\phi}{\curvearrowright} & & \\ H & \overset{\mathrm{Frob}}{\dashrightarrow} & H^{(p)} & \longrightarrow & H \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathrm{Spf}(T) & = & \mathrm{Spf}(T) & \xrightarrow{\phi} & \mathrm{Spf}(T) \end{array}$$

Definition 3.3 (Deformation of Frobenius)**.** Let (H, i, α) and (H', i', α') be two deformations of G_0 to R. A homomorphism $f\colon H \to H'$ of formal groups over R is a ***deformation of*** Frob^r if

1. $i \circ \phi^r = i'$, and

2. the following diagram commutes.

$$\begin{array}{ccccccc} G_0 & \longleftarrow & \widetilde{G_0} & \xleftarrow{\alpha} & H_0 & \longrightarrow & H \\ \downarrow{\scriptstyle \mathrm{Frob}^r} & & \downarrow{\scriptstyle \widetilde{\mathrm{Frob}^r}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ G_0^{(p^r)} & \longleftarrow & \widetilde{G_0^{(p^r)}} & \xleftarrow{\alpha'} & H_0' & \longrightarrow & H' \end{array}$$

Note that the first condition implies $\widetilde{G_0}^{(p^r)}$ is isomorphic to the base-change of G_0 along i'.

Let $\mathrm{DefFrob}_R$ be the category with objects deformations of G_0 to R and morphisms deformations of Frob^r for some $r \geqslant 0$.

If ψ and ψ' are deformations of Frob^r and $\mathrm{Frob}^{r'}$ respectively, then $\psi \circ \psi'$ (if they are composable) is a deformation of $\mathrm{Frob}^{r+r'}$.

Definition 3.4 (Graded category)**.** A ***graded category*** is a small category C together with a degree function $\deg\colon \mathrm{Mor}(C) \to \mathbb{N}$, such that $\deg(\alpha \circ \beta) = \deg(\alpha) + \deg(\beta)$ and $\deg(\mathrm{id}) = 0$.

Every morphism f in $\mathrm{DefFrob}_R$ is a deformation of Frob^r for some unique r. Thus, we can define $\deg(f) := r$. It is easy to show that $(\mathrm{DefFrob}_R, \deg)$ is a graded category.

Given $f\colon R \to R'$ in $\widehat{\mathcal{R}}$, we can define a functor $\mathrm{DefFrob}_f\colon \mathrm{DefFrob}_R \to \mathrm{DefFrob}_{R'}$ by base-change. If there is another $g\colon R' \to R''$, we have a natural isomorphism $\mathrm{DefFrob}_g \circ \mathrm{DefFrob}_f \simeq \mathrm{DefFrob}_{gf}$ satisfying the coherence condition. Hence, $\mathrm{DefFrob}\colon \widehat{\mathcal{R}} \to \mathrm{grCat}$ is a 2-functor between 2-categories.

Theorem 3.5 ([Str97, Theorem 10.1])**.** *For each $r \geqslant 0$, deformations of Frob^r is classified by an affine formal scheme $\mathrm{Spf}(L[r])$ for $L[r] \in \widehat{\mathcal{R}}$ in the following sense. For each $R \in \widehat{\mathcal{R}}$, there is an isomorphism of sets*

$$\mathrm{Hom}_{\widehat{\mathcal{R}}}(L[r], R) \simeq \{\textit{deformations of } \mathrm{Frob}^r\}/\textit{isomorphisms},$$

where two deformations of Frob^r are isomorphic if the sources and the targets are $\star$-isomorphic respectively. The isomorphism is given by base-changing of a universal deformation of Frob^r over $L[r]$ along the given homomorphism $L[r] \to R$. In particular, $L[0] \simeq E^0$.

Let $s_L\colon \mathrm{Spf}(L[r]) \to \mathrm{Def}(G_0) \simeq \mathrm{Spf}(E^0)$ and $t_L\colon \mathrm{Spf}(L[r]) \to \mathrm{Def}(G_0) \simeq \mathrm{Spf}(E^0)$ be the isogenies remembering and source and target deformations respectively. Furthermore, let $c_L\colon \mathrm{Spf}(L[r]_{t_L} \otimes_{E^0} {}_{s_L}L[r']) \to \mathrm{Spf}(L[r+r'])$ be the isogeny remembering the composition of deformations of Frob^r and $\mathrm{Frob}^{r'}$. Thus, with the structure maps s_L, t_L and c_L above, the isomorphisms in Theorem 3.5 induces an equivalence of formal (graded) category schemes,

$$\mathrm{DefFrob} \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{Spf}((\prod_{r=0}^{\infty} L[r])^{\otimes_{E^0}(-)})$$

Applying QCoh, we get the following.

Proposition 3.6 ([Rez09, 11.16])**.** *There is an equivalence of symmetric monoidal abelian* 1-*categories*

$$\begin{aligned}\mathrm{QCoh}(\mathrm{DefFrob})^{\heartsuit} &\simeq \lim_{\Delta} \mathrm{QCoh}(\mathrm{Spf}((\prod_{r=0}^{\infty} L[r])^{\otimes_{E^0}(-)}))^{\heartsuit} \\ &\simeq \lim_{\Delta_{\leqslant 2}} \mathrm{QCoh}(\mathrm{Spf}((\prod_{r=0}^{\infty} L[r])^{\otimes_{E^0}(-)}))^{\heartsuit}\end{aligned}$$

4 A modular interpretation of additive power operations

The following key theorem by Strickland relates Morava E-theory with deformation of Frobenius.

Theorem 4.1 ([Str98]). *There is an isomorphism* $\psi_r\colon L[r] \simeq A[r]$ *in* $\widehat{\mathcal{R}}$ *for each* r.

The construction of this isomorphism will be given in Section 5. In fact, there are similar structures on $A[r]$ as $\mathcal{L}[r]$.

Note that there are two E^0-algebra structure on $E^0(B\Sigma_{p^r})/I_{tr}$. One is induced by the terminal map $\Sigma^\infty_+ B\Sigma_{p^r} \to \mathbb{S}$ and the other is induced by the total additive power operation $\overline{P_{p^r}}$. We will denote by s_A, t_A for the two maps respectively. The construction of c_A needs more work. The multiplication map $\Gamma[r]_{E^0} \otimes_{E^0} \Gamma[r'] \to \Gamma[r+r']$ induces a map $A[r+r'] \to \mathrm{Hom}_{\mathrm{LMod}_{\mathrm{E}^0}}(\Gamma[r]_{E^0} \otimes_{E^0} \Gamma[r'], E^0)$. Note that we have the following isomorphism.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{LMod}_{\mathrm{E}^0}}\big(\Gamma[r'], \mathrm{Hom}_{\mathrm{LMod}_{\mathrm{E}^0}}(\Gamma[r], E^0)\big) & \longrightarrow & \mathrm{Hom}_{\mathrm{LMod}_{\mathrm{E}^0}}(\Gamma[r]_{E^0} \otimes_{E^0} \Gamma[r'], E^0) \\ f & \longmapsto & (x \otimes y) \mapsto f(y)(x) \\ y \mapsto f(- \otimes y) & \longleftarrow & f \end{array}$$

where the E^0-module structure on $\mathrm{Hom}_{\mathrm{LMod}_{\mathrm{E}^0}}(\Gamma[r], E^0)$ is given by $a \cdot f(y)(x) = f(y)(xa)$. One can check that this is the same E^0-module structure with the one induced by t_A. Since $\Gamma[r']$ is finite free over E^0, the left-hand side is isomorphic to $A[r]_{t_A} \otimes_{E^0} {}_{s_A}A[r']$. Thus, we have $c_A\colon A[r+r'] \to A[r]_{s_A} \otimes_{E^0} {}_{t_A}A[r']$ given by $f \mapsto \big((a \otimes b) \mapsto f(ab)\big)$ under the above isomorphisms.

Proposition 4.2. *The isomorphisms* $\psi_r\colon L[r] \simeq A[r]$ *in Theorem 4.1 are compatible with the morphisms* s, t *and* c.

We will postpone the proof of this proposition to the Section 5 and see what we get from it at first.

Corollary 4.3. *In particular,* $\overline{P_{p^r}}$ *corresponds to* t_L*, which is local, so is continuous for each* r.

Since $A[r]$ is dual to $\Gamma[r]$ and $\Gamma[r]$ is finite free for each r, we have

$$\mathrm{LMod}_\Gamma^\heartsuit \simeq \lim_{\Delta\leqslant 2} \mathrm{QCoh}(\mathrm{Spf}((\prod_{\mathrm{r}=0}^{\infty} \mathrm{A[r]})^{\otimes_{\mathrm{E}^0}(-)}))^\heartsuit$$

(cf. [Rez09, 5.8]). Combining this with Proposition 3.6 and Proposition 4.2, we have the following equivalences of symmetric monoidal abelian categories (cf. [Rez09, Theorem B]).

$$\mathrm{LMod}_{\Gamma}^{\heartsuit} \simeq \mathrm{QCoh}(\mathrm{DefFrob})^{\heartsuit}$$

In particular, the equivalence induces an equivalence between monoid objects. Similar to MU_*, π_0 promotes to a functor

$$\mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp_{K(n)}})) \to \mathrm{Alg}_{\Gamma} \simeq \mathrm{Alg}(\mathrm{QCoh}(\mathrm{DefFrob})^{\heartsuit}).$$

5 Translating power operations into isogenies

In this section, we will give the construction of the isomorphisms in Theorem 4.1 and prove Proposition 4.2.

If G_{univ} is the universal deformation of G_0 over E^0, then $\mathcal{O}_{G_{\mathrm{univ}}} \simeq E^0(\mathbb{CP}^\infty)$. Fix $r \in \mathbb{N}$, we aim to construct a local ring homomorphism $\psi_r \colon L[r] \to A[r]$. By Theorem 3.5, this corresponds to a deformation of Frob^r over $A[r]$. Let G_s and G_t be the formal groups over $A[r]$, which are the base-changes of G_{univ} along s_A and t_A respectively. We have $\mathcal{O}_{G_s} \simeq A[r]_{s_A} \otimes_{E^0} E^0(\mathbb{CP}^\infty)$ and $\mathcal{O}_{G_t} \simeq A[r]_{t_A} \otimes_{E^0} E^0(\mathbb{CP}^\infty)$. We have the following commutative diagram for $R \in \mathrm{CAlg}(\mathrm{Mod_E}(\mathrm{Sp_{K(n)}}))$.

$$\begin{array}{ccccccc}
\pi_0(R) & \xrightarrow{P_{p^r}} & R^0(B\Sigma_{p^r}) & \xrightarrow{\sim} & E^0(B\Sigma_{p^r}) \otimes_{E^0} \pi_0(R) & \longrightarrow & A[r]_{s_A} \otimes_{E^0} \pi_0(R) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\pi_0(E) & \xrightarrow[P_{p^r}]{} & E^0(B\Sigma_{p^r}) & = & E^0(B\Sigma_{p^r}) & \longrightarrow & A[r]
\end{array}$$

where the composite of the bottom row is t_A.

The isomorphism on the upper row is obtained from the fact that $E^0(B\Sigma_{p^r})$ is finite free over E^0. Setting $R = \mathrm{map}(\Sigma_+^\infty \mathbb{CP}^\infty, E)$ induces a local ring homomorphism

$$\mathcal{O}_{G_t} \cong A[r]_{t_A} \otimes_{E^0} E^0(\mathbb{CP}^\infty) \to A[r]_{s_A} \otimes_{E^0} E^0(\mathbb{CP}^\infty) \cong \mathcal{O}_{G_s}$$

which further gives us an isogeny $G_s \to G_t$. We claim that this isogeny is a deformation of Frob^r. Define $\psi_r \colon L[r] \to A[r]$ to be the local ring homomorphism classifying this deformation.

Lemma 5.1 ([Rez09, 10.4]). *The augmentation map $E^0(B\Sigma_{p^k}) \to E^0$ sends the transfer ideal to pE^0. Thus, it induces a map $\sigma_r \colon A[r] \to E^0/(p)$.*

By Proposition 2.7, we have the following commutative diagram.

$$\begin{array}{ccccccc}
\pi_0(R) & \xrightarrow{P_{p^r}} & R^0(B\Sigma_{p^r}) & \xrightarrow{\sim} & E^0(B\Sigma_{p^r}) \otimes_{E^0} \pi_0(R) & \longrightarrow & A[r]_{s_A} \otimes_{E^0} \pi_0(R) \\
& {\scriptstyle x \mapsto x^{p^r}} \searrow & \downarrow & & & & \downarrow {\scriptstyle \sigma_r \otimes \mathrm{id}} \\
& & \pi_0(R) & \longrightarrow & \pi_0(R)/(p) & \xleftarrow{\sim} & E^0/(p) \otimes_{E^0} \pi_0(R)
\end{array}$$

which shows that the given morphism $\mathcal{O}_{G_t} \to \mathcal{O}_{G_s}$ restricted to $E^0/(p)_{t_A} \otimes_{E^0} E^0(\mathbb{CP}^\infty) \to E^0/(p)_{s_A} \otimes_{E^0} E^0(\mathbb{CP}^\infty)$ is $[a] \otimes b \mapsto [a] \otimes b^{p^r}$, which is a deformation of Frob^r. Strickland proved that this $\psi_r \colon L[r] \to A[r]$ is an isomorphism (cf. [Str98] and [Rez09, 12.7]).

Showing that ψ_r is compatible with s, t, c is just to unravel the definition. For s, recall that $\psi_r \circ s_L$ remembers the initial deformation of the deformation of Frob^r classified by ψ_r, which is G_s. Thus, the base-changes of G_{univ} along s_A and $\psi_r \circ s_L$ are the same. By the universal property of E^0, they must equal. It is similar for t and c.

To sum up, given a $f \in \Gamma[r]$, we get a deformation of Frob^r from the base-changes of G_{univ} along $f \circ s_A$ to $f \circ t_A$ as above. Therefore, we can use isogenies between formal groups to calculate power operations explicitly.

6 Decomposition of E^0

In this section, we show an application of power operations on Morava E-theory and give an introduction to the following talk by Florian.

The recent paper [BSSW23] determines the homotopy groups of the rationalization of the $K(n)$-local sphere, which is the topic of Florian's talk.

Theorem 6.1 ([BSSW23, Theorem A]). *There is an isomorphism of graded $\mathbb{Q}_p$-algebra*

$$\pi_* L_{K(n)} \mathbb{S}[1/p] \simeq \Lambda_{\mathbb{Q}_p}(\zeta_1, \cdots, \zeta_n),$$

where the latter is the free Dirac $\mathbb{Q}_p$-algebra on generators ζ_i of degree $1-2i$.

Classically, there is an equivalence in $\mathrm{CAlg}(\mathrm{Sp}_{K(n)})$,

$$L_{K(n)} \mathbb{S} \xrightarrow{\sim} \lim_{\Delta} L_{K(n)}(E^{\otimes[-]})$$

giving by the nilpotence of $L_{K(n)}\mathbb{S}$ in $\mathrm{Mod_E(Sp_{K(n)})}$ and the $K(n)$-local Adams spectral sequence of E (cf. [DH04, Corollary A.8 and Remark A.9]). Consider the descent spectral sequence of the cosimplical object. Let X be the Dirac stack given by $\mathrm{colim}_{\Delta^{\mathrm{op}}}\mathrm{Spf}(\pi_*(E^{\otimes[-]}))$. According to [HP23, Theorem 5.5], the descent spectral sequence can be identified as

$$E_{s,t}^2 \simeq H^{-s}(X, \mathcal{O}_X(t/2)) \Rightarrow \pi_{s+t}L_{K(n)}\mathbb{S}$$

The Devinatz-Hopkins theorem gives an equivalence between Dirac stacks

$$X \simeq \mathrm{Spf}(E)/\mathbb{G}_n$$

Apply the Leray spectral sequence to $X \simeq \mathrm{Spf}(E)/\mathbb{G}_n \to B\mathbb{G}_n$, we get

$$E_{i,j}^2 \simeq H_{\mathrm{ctn}}^i(\mathbb{G}_n, H^j(\mathrm{Spf}(E), \mathcal{O}_{\mathrm{Spf}(E)}(t/2))) \Rightarrow H^{i+j}(X, \mathcal{O}_X(t/2))$$

Since $\mathrm{Spf}(E)$ is affine, the sheaf cohomology on the left-hand side is trivial for $J > 0$. Thus, the descent spectral sequence becomes the one well-known.

$$E_{s,t}^2 \simeq H_{\mathrm{cts}}^{-s}(\mathbb{G}_n, \pi_t E) \Longrightarrow \pi_{s+t}L_{K(n)}\mathbb{S}$$

However, the action of $\mathbb{G}_n$ on $\pi_*(E)$ is extremely difficult to describe. Instead, [BSSW23] considers the cohomology groups $H_{\mathrm{cts}}^*(\mathbb{G}_n, W)$. One can check that the restriction of the action of $\mathbb{G}_n$ on W is the same with the action on W by $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$. [Mor85] suggests that the natural map of $\mathbb{Z}_p$-modules,

$$H_{\mathrm{cts}}^*(\mathbb{G}_n, W) \to H_{\mathrm{cts}}^*(\mathbb{G}_n, E^0)$$

is an isomorphism after rationalization. The main result of the paper says that this conjecture is true.

Theorem 6.2 ([BSSW23, Theorem B]). *For every integer $s \geqslant 0$, the natural inclusion $W \hookrightarrow E^0$ induces a split injection*

$$H_{\mathrm{cts}}^s(\mathbb{G}_n, W) \hookrightarrow H_{\mathrm{cts}}^s(\mathbb{G}_n, E^0)$$

whose complement is p^N-torsion, where N is a positive integer independent of s. In particular,

$$H^s_{\mathrm{cts}}(\mathbb{G}_n, W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^s_{\mathrm{cts}}(\mathbb{G}_n, E^0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

Remark 6.3. *This is still open that whether this injection*

$$H^s_{\mathrm{cts}}(\mathbb{G}_n, W) \hookrightarrow H^s_{\mathrm{cts}}(\mathbb{G}_n, E^0)$$

is an isomorphism itself.

Furthermore, the linearization hypothesis by Lazard [Laz65] provides an isomorphism of Dirac $\mathbb{Q}_p$-algebras

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, W[1/p]) \simeq \Lambda_{\mathbb{Q}_p}(x_1, \cdots, x_n),$$

where $\deg(x_i) = 2i - 1$. This will finally lead towards Theorem 6.1 [BSSW23, 2.6].

The final goal of this talk is to prove the first half of Theorem 6.2, i.e.,

Proposition 6.4. *The natural inclusion $W \hookrightarrow E^0$ induces a split injection $H^s_{\mathrm{cts}}(\mathbb{G}_n, W) \hookrightarrow H^s_{\mathrm{cts}}(\mathbb{G}_n, E^0)$.*

This is an easy corollary of the following proposition.

Proposition 6.5. *The inclusion $W \hookrightarrow E^0$ admits a continuous $\mathbb{G}_n$-equivariant (additive) splitting. Equivalently, there is $\mathbb{G}_n$-equivariant decomposition of topological abelian groups $E^0 \simeq W \oplus A^c$.*

Therefore, the proof of Theorem 6.2 can be reduced to the statement that $H^*_{\mathrm{cts}}(\mathbb{G}_n, A^c)$ is p^N-torsion.

The construction of the section $E^0 \to W$ makes use of power operations in Morava E-theory. Recall from Corollary 4.3 that $\overline{P_{p^r}}$ is continuous.

Lemma 6.6 ([Rez09, 3.18]). *Suppose M is an E-module. If $G < \Sigma_k$ is a subgroup of index prime to p, then the natural map $(M^{\otimes^k_E})_{hG} \to (M^{\otimes^k_E})_{h\Sigma_k}$ admits a section.*

Proof. By Proposition A.1, the composition $(M^{\otimes^k_E})_{h\Sigma_k} \overset{\mathrm{tr}^{\Sigma_k}_G}{\to} (M^{\otimes^k_E})_{hG} \to (M^{\otimes^k_E})_{h\Sigma_k}$ is a multiplication by the index of G in Σ_k, which is prime to p. Since E is p-local, the composition is an isomorphism. □

Lemma 6.7. *The total power operations on Morava E-theory are continuous with respect to the $\mathfrak{m}$-adic topology on E^0 and $E^0(B\Sigma_k)$.*

Proof. Assume that $k = \sum_{i=0}^{m} a_i p^i$ is the base-p expansion of m. By Lemma 6.6, we have that the natural map induced by inclusion of subgroups $\prod_{i=0}^{m} \Sigma_{p^i}^{\times a_i} \hookrightarrow \Sigma_k$

$$E^0(B\Sigma_k) \to E^0(B(\prod_{i=0}^{m} \Sigma_{p^i}^{\times a_i})) \simeq \bigotimes_{i=0}^{m} {}_{E^0} E^0(B\Sigma_{p^i})^{\otimes_{E^0} a_i}$$

is injective, as $\prod_{i=0}^{m} \Sigma_{p^i}^{\times a_i} < \Sigma_k$ is a subgroup of index prime to p. The equivalence here is given by Proposition 2.9 and the Kunneth formula. Since W is a DVR, E^0 is Noetherian. The Artin-Rees lemma implies that the inclusion is an embedding of topological groups. Thus, it suffices to show that the composition

$$E^0 \xrightarrow{P_k} E^0(B\Sigma_k) \hookrightarrow \bigotimes_{i=0}^{m} {}_{E^0} E^0(B\Sigma_{p^i})^{\otimes_{E^0} a_i}$$

is continuous. Note that this map can be identified as $\bigotimes (P_{p^i})^{\otimes a_i}$. Thus, it remains to show that P_{p^i} is continuous. For each $0 \leqslant j \leqslant i$, we have a natural map of E^0-algebras.

$$E^0(B\Sigma_{p^i}) \to \left(E^0(B\Sigma_{p^j})\right)^{\otimes_{E^0} p^{i-j}} \to \left(E^0(B\Sigma_{p^j})/I_{tr}\right)^{\otimes_{E^0} p^{i-j}},$$

where the first map is induced by inclusion of groups $\Sigma_{p^j}^{\times p^{i-j}} \hookrightarrow \Sigma_{p^k}$ and the latter map is taking quotient by transfer ideals. The composition of this map with P_{p^i} can be identified as $(\overline{P_{p^j}})^{\otimes p^{i-j}}$, which is continuous. Now taking the product of all such j,

$$E^0(B\Sigma_{p^i}) \to \prod_{j=0}^{i} \left(E^0(B\Sigma_{p^j})/I_{tr}\right)^{\otimes_{E^0} p^{i-j}}$$

We claim that the Hopkins-Kuhn-Ravenel character theory implies that this map is injective (Appendix B). The composition of this map and P_{p^i} is continuous. By applying the above trick again, we get P_{p^i} is continuous. □

Proof for Proposition 6.5. For each m, let

$$\beta_m \colon E^0 \xrightarrow{P_m} E^0(B\Sigma_m) \xrightarrow{\widehat{\mathrm{tr}}^e_{\Sigma_m}} E^0,$$

where $\widehat{\mathrm{tr}}^{e}_{\Sigma_m}$ is the $K(n)$-local transfer map along the surjection $\Sigma_m \to e$, which is defined in Appendix A. Thus, β_m is continuous by the above lemma.

Define $\beta\colon E^0 \to E^0[\![x]\!]$ be the formal sum $a \mapsto \sum_{m=0}^{\infty} \beta_m(a)x^m$. We have the relationship $\beta(a+b) = \beta(a)\beta(b)$, which is proved at the end of Appendix A. Furthermore, $\beta_0(a) = 1$ and $\beta_1(a) = a$ for all $a \in E^0$. Thus, β factors through as a continuous group homomorphism $E^0 \to 1 + E^0[\![x]\!]$. Now we take the quotient of the target by $1 + E^0[\![x]\!] \to 1 + \overline{\mathbb{F}_p}[\![x]\!]$. Post-compose it with the map $1 + \overline{\mathbb{F}_p}[\![x]\!] \to \overline{\mathbb{F}_p}$ given by reading off the coefficient of x. The map factors through W since $1 + \overline{\mathbb{F}_p}[\![x]\!] \cong \mathbb{W}_{\mathrm{big}}(k)$ canonically as abelian groups (Warning: they are not canonically isomorphic as rings), and the map can be identified as the projection to the first component of $\mathbb{W}_{\mathrm{big}}(k)$ under the isomorphism. This projection map factors through the restriction map $\mathbb{W}_{\mathrm{big}}(k) \to W$. We can summarize these in the following diagram.

$$\begin{tikzcd}
W \arrow[rr, "f"] \arrow[d, hook] & & W \arrow[r] & \overline{\mathbb{F}_p} \\
 & & \mathbb{W}_{\mathrm{big}}(k) \arrow[u] & \\
E^0 \arrow[r, "\beta"] \arrow[rru, "\gamma"] & 1 + E^0[\![x]\!] \arrow[r] & 1 + \overline{\mathbb{F}_p}[\![x]\!] \arrow[u, "\sim"'] \arrow[ruu] &
\end{tikzcd}$$

Let $\gamma\colon E^0 \to W$ be the composition of the maps above as in the diagram, which is continuous. Pre-composing γ with the inclusion $W \hookrightarrow E^0$, we have an additive endomorphism f of W. Note that f is the identity of $\overline{\mathbb{F}_p}$ modulo p since $\beta_1 = \mathrm{id}$, so f is an isomorphism by the universal property of W. Hence, $\alpha := f^{-1} \circ \gamma$ is the required continuous splitting.

It remains to show that α is $\mathbb{G}_n$-equivariant. Since $\mathbb{G}_n = \mathrm{Aut}_{\mathrm{CAlg}(\mathrm{Sp}_{K(n)})}(E)$, P_m is $\mathbb{G}_n$-equivariant. Since $\widehat{tr}^{e}_{\Sigma_m}$ is a map induced from spectra level, β_m is $\mathbb{G}_n$-equivariant. Note that both the inclusion $W \hookrightarrow E^0$ and the projection $E^0 \to \overline{\mathbb{F}_p}$ are $\mathbb{G}_n$-equivariant. Therefore, α is equivariant. $\square$

A Transfer maps

Suppose G is a finite group and $H < G$. Let $i\colon BH \to BG$ be the natural map induced by the inclusion $H < G$. In the six-functor formalism of anima, we have

$$\begin{tikzcd} \mathrm{Sp}^{BG} \arrow[r, "i^*\simeq i^!"] & \mathrm{Sp}^{BH} \arrow[l, bend right=60, "i_!"', "\perp"] \arrow[l, bend left=60, "i_*", "\perp"'] \end{tikzcd}$$

Note that the fiber of i is G/H, which is finite and static, so i is proper. Thus, the norm map $\mathrm{Nm}_i\colon i_! \to i_*$ is an equivalence.

Let $p\colon BG \to 1$ and $q\colon BH \to 1$ be the terminal maps. The map

$$\mathrm{tr}_H^G\colon p_! \to p_! i_* i^* \xleftarrow{\simeq} p_! i_! i^! \simeq q_! i^*$$

is called the ***transfer map***.

Proposition A.1 (cf. [BMMS86, II.1.11])**.** *The composition* $p_! \xrightarrow{\mathrm{tr}_H^G} p_! i_! i^! \to p_!$ *induces a multiplication by* $|G|/|H|$ *on homotopy groups.*

Proposition A.2. *If* $K < H < G$*, then* $\mathrm{tr}_K^H \circ \mathrm{tr}_H^G \simeq \mathrm{tr}_K^G$ *to be an equivalence to be specified in the proof.*

Proof. We have a commutative diagram

$$\begin{tikzcd} BK \arrow[r, "j"] \arrow[rr, bend left, "k"] \arrow[dr, "r"'] & BH \arrow[r, "i"] \arrow[d, "q"] & BG \arrow[dl, "p"] \\ & 1 & \end{tikzcd}$$

induced by the group inclusions and the terminal maps. We have the following diagram given by the natural maps.

$$\begin{tikzcd} p_! \arrow[r] & p_! i_* i^* \arrow[d] & p_! i_! i^* \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & q_! i^* \arrow[d] & & \\ & p_! i_* j_* j^* i^* \arrow[d, "\sim"'] & p_! i_! j_* j^* i^* \arrow[l, "\sim"'] \arrow[r, "\sim"] & q_! j_* j^* i^* & q_! j_! j^* i^* \arrow[l, "\sim"'] \arrow[d, "\sim"] & \\ & p_! k_* k^* & & & p_! k_! k^* \arrow[lll, "\sim"'] \arrow[r, "\sim"] & r_! k^* \end{tikzcd}$$

where the top right route is $\mathrm{tr}_K^H \circ \mathrm{tr}_H^G$ and the left bottom route is tr_K^G. □

Now we turn to the $K(n)$-local transfer map $\widehat{\mathrm{tr}}_\Sigma^e$ mentioned in the proof for Proposition 6.5. Let $p\colon BG \to 1$ as above. We have

$$\begin{tikzcd} (\mathrm{Sp}_{K(n)})^{*} \arrow[r, "p^*\simeq p^!" description] & (\mathrm{Sp}_{K(n)})^{BG} \arrow[l, bend right, "p_!"', "\perp"] \arrow[l, bend left, "p_*", "\perp"'] \end{tikzcd}$$

By [HL13, Theorem 5.2.1], which is called 'ambidexterity in $K(n)$-local spectra', the norm map $\mathrm{Nm}_p\colon p_! \xrightarrow{\sim} p_*$ is an isomorphism. The map

$$\widehat{\mathrm{tr}}^{e}_{G}\colon \mathrm{Id}_{\mathrm{Sp}_{K(n)}} \to p_*p^* \xleftarrow{\sim} p_!p^*$$

is called the $K(n)$***-local transfer map***. Putting X to be $L_{K(n)}\mathbb{S}$, we get the transfer map as in the proof.

Furthermore, we claim in the proof that $\beta(a+b)=\beta(a)\beta(b)$, or equivalently, $\beta_k(a+b)=\sum_{i=0}^{k}\beta_i(a)\beta_{k-i}(b)$ for all k.

Lemma A.3. *If G, H are static groups, then $\widehat{\mathrm{tr}}^{e}_{G\times H}(L_{K(n)}\mathbb{S}) \simeq \widehat{\mathrm{tr}}^{e}_{G}(L_{K(n)}\mathbb{S}) \otimes \widehat{\mathrm{tr}}^{e}_{H}(L_{K(n)}\mathbb{S})$ by an equivalence to be specified in the proof.*

Proof. Denote the terminal maps and the projection maps as follows.

$$\begin{tikzcd} BG\times BH \arrow[r, "q'"] \arrow[d, "p'"] \arrow[dr, "p\times q"] & BG \arrow[d, "p"] \\ BH \arrow[r, "q"] & 1 \end{tikzcd}$$

We want to show that the following diagram commutes, where the $\widehat{\otimes}$ denotes for the tensor product in $\mathrm{Sp}_{K(n)}$.

$$\begin{tikzcd} L_{K(n)}\mathbb{S} \arrow[r] & (p\times q)_*(p\times q)^*(L_{K(n)}\mathbb{S}) & (p\times q)_!(p\times q)^*(L_{K(n)}\mathbb{S}) \arrow[l, "\mathrm{Nm}_{p\times q}"'] \\ L_{K(n)}\mathbb{S}\widehat{\otimes}L_{K(n)}\mathbb{S} \arrow[u, "\sim"] \arrow[r] & p_*p^*(L_{K(n)}\mathbb{S})\widehat{\otimes}q_*q^*(L_{K(n)}\mathbb{S}) \arrow[u] & p_!p^*(L_{K(n)}\mathbb{S})\widehat{\otimes}q_!q^*(L_{K(n)}\mathbb{S}) \arrow[u] \arrow[l, "\mathrm{Nm}_p\widehat{\otimes}\mathrm{Nm}_p"'] \end{tikzcd}$$

The left square commutes because $L_{K(n)}\mathbb{S}$ is the tensor unit in $\mathrm{Sp}_{K(n)}$. The right square commutes because we can expand the vertical maps using the projection formula and the Beck-Chevalley maps as follows.

$$\begin{array}{ccc}
(p\times q)_*(p\times q)^*(L_{K(n)}\mathbb{S}) & \longleftarrow & (p\times q)_!(p\times q)^* \\
\uparrow & & \uparrow \\
(p\times q)_*((p\times q)^*\widehat{\otimes}(p\times q)^*) & \longleftarrow & (p\times q)_!((p\times q)^*\widehat{\otimes}(p\times q)^*) \\
\uparrow & & \downarrow\wr \\
p_*(p^*\widehat{\otimes}q'_*p'^*q^*) & \longleftarrow & p_!(p^*\widehat{\otimes}q'_!p'^*q^*) \\
\uparrow\wr & & \downarrow\wr \\
p_*(p^*\widehat{\otimes}p^*q_*q^*) & \longleftarrow & p_!(p^*\widehat{\otimes}p^*q_!q^*) \\
\uparrow & & \downarrow\wr \\
p_*p^*\widehat{\otimes}q_*q^* & \longleftarrow & p_!p^*\widehat{\otimes}q_!q^*
\end{array}$$

where the horizontal maps are norms maps from lower shrieks to lower stars. Each small diagram commutes by routine check. □

Now we compute

$$\begin{aligned}
\beta_k(a+b) &= (\widehat{\mathrm{tr}}^{e}_{\Sigma_k})^* P_k(a+b) \\
&= \sum_{i=0}^{k} (\widehat{\mathrm{tr}}^{e}_{\Sigma_k})^* (\widehat{\mathrm{tr}}^{\Sigma_k}_{\Sigma_i\times\Sigma_{k-i}})^* (P_i(a)P_{k-i}(b)) \\
&= \sum_{i=0}^{k} (\widehat{\mathrm{tr}}^{\Sigma_k}_{\Sigma_i\times\Sigma_{k-i}} \circ \widehat{\mathrm{tr}}^{e}_{\Sigma_k})^* (P_i(a)P_{k-i}(b)) \\
&= \sum_{i=0}^{k} (\widehat{\mathrm{tr}}^{e}_{\Sigma_i\times\Sigma_{k-i}})^* (P_i(a)P_{k-i}(b)) \\
&= \sum_{i=0}^{k} \big((\widehat{\mathrm{tr}}^{e}_{\Sigma_i})^* P_i(a)\big)\big((\widehat{\mathrm{tr}}^{e}_{\Sigma_{k-i}})^* P_{k-i}(b)\big) \\
&= \sum_{i=0}^{k} \beta_i(a)\beta_{k-i}(b)
\end{aligned}$$

The fourth equality is similar to Proposition A.2 and the fifth equality is by the above lemma.

B A very brief introduction to HKR character theory

In this section, we give a very brief introduction to HKR character theory, in order to give a proof for the claim at the end of the proof for Lemma 6.7. For a detailed explanation of HKR character theory, one may read Maxime's notes [Ram].

Fix a finite group G. Recall the Atiyah-Segal completion theorem says that there is an isomorphism $\mathrm{KU}^0(BG) \simeq R(G)^{\wedge}_{I(G)}$, where $R(G)$ is the complex representation ring of G and $I(G)$ is the augmentation ideal. This gives an interpretation of $\mathrm{KU}^0(BG)$ in terms of class functions. Note that at height 1, Morava E-theory is equivalent to $\mathrm{KU}^{\wedge}_p$. The idea of the HKR character theory is a chromatic (p-local) generalization of the Atiyah-Segal completion theorem to higher heights.

More specifically, let $G_p^n := \mathrm{Hom}(\mathbb{Z}_p^n, G)$, where G acts on G_n^p by conjugation. For each compact G-anima X, the HKR character theory gives us a character map

$$\Phi_G \colon E^0(X_{hG}) \to C_0 \otimes_{p^{-1}E^0} (p^{-1}E)^0((\coprod_{\alpha \in G_p^n} X^{\mathrm{im}(\alpha)})_{hG}),$$

where C_0 is the smallest ring extension of $p^{-1}E^0$ such that the induced

$$C_0 \otimes_{E^0} E^0(X_{hG}) \to C_0 \otimes_{p^{-1}E^0} (p^{-1}E)^0((\coprod_{\alpha \in G_p^n} X^{\mathrm{im}(\alpha)})_{hG})$$

is an isomorphism.

Proposition B.1 (cf. [Ram, Proposition 2.30 and Remark 2.33])**.** *C_0 is a flat extension of $p^{-1}E^0$.*

Corollary B.2. *Φ_G is injective.*

In particular, if we take $X = *$, then $X_{hG} \simeq BG$ and $\coprod_{\alpha \in G_p^n} X^{\mathrm{im}(\alpha)} \simeq G_p^n$, so that $(G_p^n)_{hG} \simeq \coprod_{[\alpha] \in G_p^n/G} BC_G(\mathrm{im}(\alpha))$, where C_G denotes the centralizer in G. Since $p^{-1}E$ is rational, $(p^{-1}E)^0(BG) \simeq (p^{-1}E)^0$ similar to Lemma 6.6. Thus, Φ_G can be identified as an injection

$$\Phi_G \colon E^0(BG) \hookrightarrow \prod_{G_n^p/G} C_0$$

Now we want to show that

$$E^0(B\Sigma_{p^i}) \overset{\prod (\overline{P_{p^j}})^{\otimes p^{i-j}}}{\longrightarrow} \prod_{j=0}^{i} \big(E^0(B\Sigma_{p^j})/I_{tr}\big)^{\otimes_{E^0} p^{i-j}}$$

is injective at the end of the proof of Lemma 6.7.

According to [Ram, Proposition 3.10], Φ_G is compatible with group inclusions. That is, we have the following commutative diagram.

$$\begin{array}{ccc}
E^0(B\Sigma_{p^i}) & \xhookrightarrow{\Phi_{\Sigma_{p^i}}} & \prod_{(\Sigma_{p^i})^n_p/\Sigma_{p^i}} C_0 \\
\downarrow & & \downarrow \\
\prod_{j=0}^i E^0(B\Sigma_{p^j})^{\otimes_{E^0} p^{i-j}} & \xhookrightarrow{\Phi_{\Sigma_{p^j}^{\times p^{i-j}}}} & \prod_{j=0}^i \prod_{(\Sigma_{p^j}^{\times p^{i-j}})^n_p/\Sigma_{p^j}^{\times p^{i-j}}} C_0
\end{array}$$

where the right vertical map is given as the following. We have a morphism between indexes $(\Sigma_{p^j}^{\times p^{i-j}})^n_p/\Sigma_{p^j}^{\times p^{i-j}} \to (\Sigma_{p^i})^n_p/\Sigma_{p^i}$ induced from the group inclusion $\Sigma_{p^j}^{\times p^{i-j}} \hookrightarrow \Sigma_{p^i}$. The vertical map is given by projecting each component in $\prod_{(\Sigma_{p^i})^n_p/\Sigma_{p^i}} C_0$ to the components indexed by the preimages of the index under the above map between indexes.

Again, by [Ram, Proposition 3.10], we have another commutative diagram.

$$\begin{array}{ccc}
E^0(B\Sigma_k \otimes B\Sigma_{p^j-k}) & \xhookrightarrow{\Phi_{\Sigma_k \times \Sigma_{p^j-k}}} & \prod_{(\Sigma_k \times \Sigma_{p^j-k})^n_p/(\Sigma_k \times \Sigma_{p^j-k})} C_0 \\
\Big\downarrow (\mathrm{tr}_{\Sigma_k \times \Sigma_{p^j-k}}^{\Sigma_{p^j}})^* & & \downarrow \\
E^0(B\Sigma_{p^j}) & \xhookrightarrow[\Phi_{\Sigma_{p^j}}]{} & \prod_{(\Sigma_{p^j})^n_p/\Sigma_{p^j}} C_0
\end{array}$$

where the right vertical map is also induced from the canonical map between indexes, but this time we map components indexed by the preimages of an index to the image component.

Therefore, we have the following diagram.

$$\begin{array}{ccc}
E^0(B\Sigma_{p^i}) & \xhookrightarrow{\Phi_{\Sigma_{p^i}}} & \prod_{(\Sigma_{p^i})^n_p/\Sigma_{p^i}} C_0 \\
\downarrow & & \downarrow \quad \iota \\
\prod_{j=0}^i E^0(B\Sigma_{p^j})^{\otimes_{E^0} p^{i-j}} & \xhookrightarrow{\prod_{j=0}^i \Phi_{\Sigma_{p^j}^{\times p^{i-j}}}} & \prod_{j=0}^i \prod_{(\Sigma_{p^j}^{\times p^{i-j}})^n_p/\Sigma_{p^j}^{\times p^{i-j}}} C_0 \\
\downarrow & & \downarrow \\
\prod_{j=0}^i \left(E^0(B\Sigma_{p^j})/I_{tr}\right)^{\otimes_{E^0} p^{i-j}} & \longrightarrow & \prod_{j=0}^i \prod_{(\Sigma_{p^j}^{\times p^{i-j}})^n_p/\Sigma_{p^j}^{p^{i-j}} - \coprod_{k=1}^{p^j-1}((\Sigma_k \times \Sigma_{p^j-k})^{\times p^{i-j}})^n_p/(\Sigma_k \times \Sigma_{p^j-k})^{\times p^{i-j}}} C_0
\end{array}$$

It suffices to show that the map ι in the diagram is injective, which implies that the composition of maps in the left column is injective. Explicitly, ι is a map induced from a map between indexes by mapping the image component to the preimage components. If we can show that the map between indexes is surjective, it will imply ι is injective. Running through the above process, the map between indexes is given by

$$\coprod_{j=0}^{i} \left((\Sigma_{p^j}^{\times p^{i-j}})^n_p/\Sigma_{p^j}^{p^{i-j}} - \coprod_{k=1}^{p^j-1} ((\Sigma_k \times \Sigma_{p^j-k})^{\times p^{i-j}})^n_p/(\Sigma_k \times \Sigma_{p^j-k})^{\times p^{i-j}} \right) \to (\Sigma_{p^i})^n_p/\Sigma_{p^i}$$

induced from group inclusions. Note that

$$G_p^n = \{\sigma = (\sigma_1, \cdots, \sigma_n) \colon \sigma_m \in G \text{ is } p\text{-power torsion and } \sigma_l \sigma_m = \sigma_m \sigma_l \text{ for all } l, m\}$$

Given any $\sigma \in (\Sigma_{p^i})_p^n$, σ_m can be factorized as a multiplication of permutations on disjoint cycles for each i. Since σ_m is p-power torsion, each cycle has length of a power of p. Suppose the maximal length of cycles in σ_m if p^{l_m}. By conjugation, we can assume that $\sigma_m \in \Sigma_{\Sigma_{p^{l_m}}^{p^{i-l_m}}} < \Sigma_{p^i}$. Suppose $l := \max_{1 \leqslant m \leqslant n}\{l_m\}$. Since σ_m are commuting, they can be simultaneously conjugate to some elements in $\Sigma_{p^l}^{p^{i-l}}$. Thus, σ is in the image of $(\Sigma_{p^l}^{p^{i-l}})_p^n/\Sigma_{p^l}^{p^{i-l}} \to (\Sigma_{p^i})_p^n/\Sigma_{p^i}$.

Since there is a cycle in σ with length l, the preimage of σ in $(\Sigma_{p^l}^{p^{i-l}})_p^n/\Sigma_{p^l}^{p^{i-l}}$ is not in the image of $((\Sigma_k \times \Sigma_{p^l-k})^{\times p^{i-l}})_p^n/(\Sigma_k \times \Sigma_{p^l-k})^{\times p^{i-l}} \to (\Sigma_{p^l}^{\times p^{i-l}})_p^n/\Sigma_{p^l}^{p^{i-l}}$ for any $1 \leqslant k < l$. Thus, the map between indexes is surjective.

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